Soliton-potential interaction in the ϕ^4 model

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A model for the soliton-potential scattering is presented. This model is constructed with a better approximation for adding the potential to the Lagrangian through the metric of background space time. The results of the model are compared with other models, and the differences are discussed.

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I. INTRODUCTION

Scattering of solitons from potentials have been studied in many papers by different methods. It is shown that the soliton acts as a pointlike particle in most of the cases. But there are some surprising features which need more attention. Solitons interact on a potential barrier almost elastically; while they show quantumlike behavior during the interaction with a potential well. Several models have been presented to explain such behavior. Models are different in the method of adding the potential to the soliton equation of motion. The general behavior of the solitons in all the models is the same, but details of the scattering and some interesting features are different. For example, some of the models predict that a soliton may reflect backward (or forward) after the scattering on a potential well while other models do not have this behavior. The potential can be added to the equation of motion as a perturbative term [1,2]. These effects also can be taken into account by making some parameters of the equation of motion to be function of space or time [3]. Also one can add such effects to the Lagrangian of the system by introducing a suitable nontrivial metric for the background space time, without missing the topological boundary conditions. This method has been used for studying the sine-Gordon model in [4,5].

Scattering of ϕ^4 solitons on barriers and holes has been investigated with two different models in [6]. In one of the models, the potential has been considered by deforming one of the parameters of the soliton equation of motion. In another model the same potential has been added through the metric of background space time. Some differences between the results of two models have been reported in this paper. Also there are other important differences between the models which need attention. There are some important questions: what is the reason for the differences between the results of two models? Which model makes a better explanation for the nature of the system? Is it possible to find a better approximation for one of the models in order to have agreement between the models? If the answer is yes then we can conclude that two models are well. Note that these two models rise from different approaches. This investigation will give us a brighter situation from the soliton-potential scattering.

The answer is important from the other point of view. The results can show us which behavior are model independent and completely come from the physics of solitons. Results of a model are more believable if we can derive them from other models which are constructed with different manners. In this situation we can tune the parameters of the models in order that all the models show the same behavior in detail. Afterward counterpart parameters of different models can be compared. Such a condition gives us sharper knowledge about the effects of the modeling methods on the results of an investigation on nonlinear systems. Similarities and differences of three models are inspected in this paper. We will try to explain the features of the models in order to find a better description for the system.

The results also are very important for constructing suitable collective coordinate variables in the modeling of nonlinear systems. The metric model has been used in constructing a collective coordinate system for topological solitons in [7]. Therefore if we find a better approximation for this model then we can improve the related models and their results.

Motivated by these questions we studied the similarities and the differences between the models. A brief description of the models and their results is presented in Sec. II. A better approximation for one of the models is presented in Sec. III. A new model is used for investigating the soliton-potential interaction in ϕ^4 model and the results are compared with the results of the other models in Sec. IV. Some conclusions and remarks will be presented in Sec. V.

II. TWO MODELS FOR "ϕ^4 SOLITONS-POTENTIAL" SYSTEM

Model 1. Consider a scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi) \tag{1}$$

and the following potential:

$$U(\phi) = \lambda(x)(\phi^2 - 1)^2. \tag{2}$$

The equation of motion for the field becomes

$$\partial_\mu \partial^\mu \phi + 4\lambda(x)\phi(\phi^2 - 1) = 0. \tag{3}$$

The effects of the potential are added to the equation of motion by using a suitable definition for $\lambda(x)$, like $\lambda(x)=1+v(x)$. For a constant value of parameter $(\lambda(x)=1)$ Eq. (2) has a solitary solution as following [6]:

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$$\phi(x,t) = \pm \tanh\left(\frac{\sqrt{2}(x-x_0-ut)}{\sqrt{1-u^2}}\right) \quad (4)$$

in which x_0 and u are solitary wave initial position and its initial velocity, respectively. This equation is used as an initial condition for solving Eq. (3) with a space dependent $\lambda(x)$ when the potential $v(x)$ is small.

Model 2. The potential also can be added to the Lagrangian of the system, through the metric of background space time. So the metric includes characteristics of the medium. The general form of the action in an arbitrary metric is

$$I = \int \mathcal{L}(\phi,\partial_\mu\phi)\sqrt{-g}d^n x dt \quad (5)$$

where "g" is the determinant of the metric $g^{\mu\nu}(x)$. A suitable metric in the presence of a weak potential $v(x)$ is [4–6],

$$g^{\mu\nu}(x) \cong \begin{pmatrix} 1+V(x) & 0 \\ 0 & -1 \end{pmatrix}. \quad (6)$$

The equation of motion for the field ϕ which is described by the Lagrangian (1) in the action (5) is [4,7]

$$\frac{1}{\sqrt{-g}}(\sqrt{-g}\partial_\mu\partial^\mu\phi + \partial_\mu\phi\partial^\mu\sqrt{-g}) + \frac{\partial U(\phi)}{\partial\phi} = 0. \quad (7)$$

This equation of motion in the background space-time (6) becomes [6]

$$(1+V(x))\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{1}{2|1+V(x)|}\frac{\partial V(x)}{\partial x}\frac{\partial\phi}{\partial x} + \frac{\partial U(\phi)}{\partial\phi} = 0. \quad (8)$$

The field energy density is

$$\mathcal{H}_2 = g^{00}(x)\left(\frac{1}{2}g^{00}(x)\dot{\phi}^2 + \frac{1}{2}\acute{\phi}^2 + U(\phi)\right). \quad (9)$$

The energy density is calculated by varying both the field and the metric (see page 643 Eq. (11.81) of [8]). If we look at the $\mathcal{L}(\phi,\partial_\mu\phi)\sqrt{-g}$ of action (5) as an effective Lagrangian in a flat space-time, then the energy density becomes [6]

$$\epsilon = \sqrt{g^{00}(x)}\left(\frac{1}{2}g^{00}(x)\dot{\phi}^2 + \frac{1}{2}\acute{\phi}^2 + U(\phi)\right). \quad (10)$$

The above Hamiltonian density can be found by varying only the field in the effective Lagrangian. This equation does not contain energy exchange between the field and the metric (therefore with the potential). This is an important point that makes some difference.

Solution (4) can be used as an initial condition for solving Eq. (8) when the potential $v(x)$ is small. If we define the parameter $\lambda(x)=1+v(x)$ in model 1 then it is possible to compare the results of two models. The potential

$$\lambda = \begin{Bmatrix} \lambda_0 & |x| \le p \\ 0 & |x| > p \end{Bmatrix} \quad (11)$$

has been chosen in [4] where λ_0 and "p" are the potential strength and the potential width, respectively.

Scattering of a soliton with a potential barrier is nearly elastic. The soliton radiates a small amount of energy during the interaction. The radiated energy during the interaction in model 2 is more than the energy radiation in the model 1. The radiated energy becomes larger when the height of the barrier or the speed of the soliton increases [4].

The two models show that there exist two different kinds of trajectories during the scattering of a soliton on a potential barrier depending on soliton initial velocity. Two kinds of trajectories are separated by a critical velocity u_c. At low velocities ($u<u_c$) soliton reflects back and reaches its initial place. A soliton with an initial velocity $u>u_c$ has enough energy to pass over the potential.

Note that model 2 is valid for slowly varying small potentials [4]. It means that we have to use smooth and small potentials in metric (6), but a squarelike potential is not smooth. Some simulations have been set up with a smooth potential $v(x)=ae^{-b(x-c)^2}$ with using two models. Simulations show that the differences between the soliton behavior in two models with using this potential decrease, but the differences do not vanish. Indeed the simulations with this potential are in agreement with the mentioned differences which have been reported for a squarelike potential in [6]. It means that the two models are really different in these cases.

Is it possible to improve a model in order to find agreement with the other model?

III. IMPROVED APPROXIMATION

It is possible to improve model 2 by using a better approximation for the metric of back ground space time. Model 2 has been constructed using classical limit of a potential in general relativity with the first term of approximation [4]. We can improve the model with adding the second term of approximation to the metric of space-time. Therefore the better metric is

$$g^{\mu\nu}(x) \cong \begin{pmatrix} 1+V(x) & 0 \\ 0 & -\dfrac{1}{1+v(x)} \end{pmatrix}. \quad (12)$$

The field equation of motion is derived from Eq. (7) using the metric (12) as follows:

$$[1+V(x)]\frac{\partial^2\phi}{\partial t^2} - \frac{1}{1+v(x)}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial U(\phi)}{\partial\phi} = 0. \quad (13)$$

The energy density of the field in this improved model (which we call it "model 3") becomes

$$\mathcal{H}_3 = [g^{00}(x)]^2\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\acute{\phi}^2 + g^{00}(x)U(\phi). \quad (14)$$

The potential is still a slowly varying and small function.

The models can be compared using the energy of a soliton in three models. The energy density of a soliton in model 1 is

$$\mathcal{H}_1 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\acute{\phi}^2 + g^{00}(x)U(\phi). \quad (15)$$

The above energy density is related to the field only. Equations (9) and (14) are the energy density of "soliton

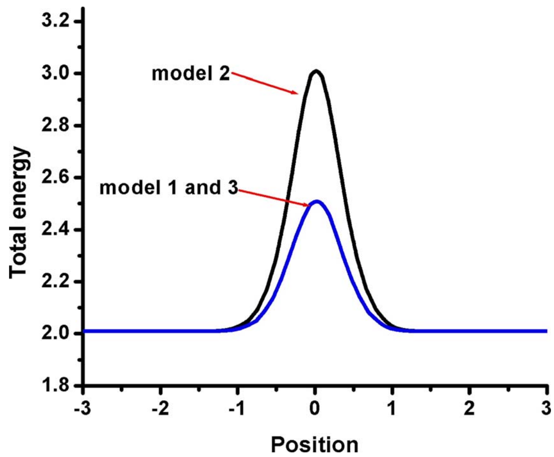

FIG. 1. (Color online) Potential barrier $v(x)=0.5e^{-4x^2}$ as seen by the soliton in three models.

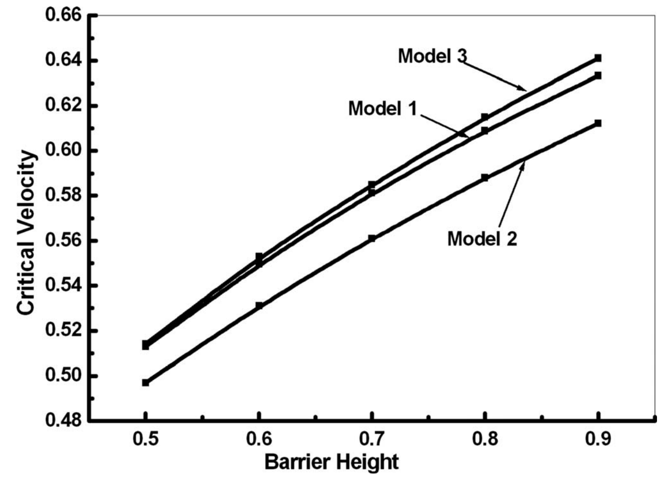

FIG. 2. Critical velocity as a function of the barrier height of the potential $v(x)=0.5e^{-4x^2}$ in three models.

+metric" in the model 2 and the new model 3, respectively. Consider a static solitary wave located in the initial position X_0. The energy density of the field is $\frac{1}{2}\acute{\phi}^2+[1+v(x)]U(\phi)$ in the models 1 and 3 while the model 2 gives the static energy $\frac{1}{2}[1+v(x)]\acute{\phi}^2+[1+v(x)]U(\phi)$ for this situation. The static energy difference between models 2 and 3 (or 1) is $\Delta E_{static}=E_{model2}-E_{model3}=\frac{1}{2}v(x)\acute{\phi}^2$. It means that the models 1 and 3 have the same effective potential but the model 2 contains an extra term. This small extra term is positive for a potential barrier and it is negative for potential well. It can be concluded that the effective potential in the model 2 is stronger than the effective potential of models 1 and 3. Let us now look at the kinetic terms in three models. The kinetic energy in models 2 and 3 are the same and they are different with the kinetic energy in the model 1. The difference kinetic energy in the first-order approximation is $\Delta E_{kinetic}=E_{Kmodel3}-E_{Kmodel1}\cong v(x)\dot{\phi}^2$. It is positive for potential barrier and negative for the potential well. Thus the effective mass in the model 1 is smaller than the effective mass of the soliton in the models 2 and 3 for a potential barrier. These differences act on the dynamics of solitary waves in the opposite ways therefore the predictions of the model 3 will be something between the predictions of the other models. The effects of these differences are studied in the next section.

IV. COMPARING THE MODELS

Several simulations using three models have been performed with different types of potentials. The potential $v(x)=ae^{-b(x-c)^2}$ has been used in simulations which are reported below. This type of potential is more suitable than a rectangular potential because it is a smooth and slowly varying function.

Figure 1 shows the shape of the potential barrier $v(x)=0.5e^{-4x^2}$ as seen by the soliton in three models. The shape of this potential has been found by placing a static soliton at different positions and calculating its total energy. As we have seen, the static energy of the models 1 and 3 are equal while the static energy calculated using model 2 is different. Our calculations are not in agreement with the simulations of Ref. [6]. It is because of the difference in the way of calculating the Hamiltonian density. The calculated energy density in [6] contains only the energy of the soliton. Varying of the metric did not included in the calculation of the energy density, because the potential has been taken fixed. Note that the Hamiltonian density (9) is calculated with varying the field and metric. In other words, energy exchange between field and metric is possible.

The critical velocity of a soliton to pass over the potential barrier has been demonstrated as a function of the barrier height in Fig. 2 for three models. There are some differences between the results of the models. The simulation results of models 1 and 3 are more similar and they are different from the results of model 2. Varying of the metric was not included in the calculation of the energy density in [6], because the potential has been taken fixed. Therefore calculated energy density [Eq. (10)] contains only the energy of the soliton. Note that the Hamiltonian density (9) is calculated with varying the field and the metric. Equations (9) and (10) are different in the second order of potential magnitude. However this difference does not change the general behavior of the system, but affects on the details of the interaction; for example on the values of effective potential and its width, the amount of radiated energy during the interaction and so on. Differences are noticeable in a soliton-well system which will be discussed later. Figure 3 shows the critical velocity as a function of the potential width for three models. This figure shows very good agreement between models 1 and 3 too.

It is mentioned that the static energy of a soliton calculated using the models 1 and 3 are less than the calculated energy in the model 2. Simulations also show that the static energy of a soliton at the peak of the barrier calculated with the models 1 and 3 is the same and they are less than the static energy calculated using model 2 (See Fig. 1). Figures 2 and 3 show that the critical velocity simulated using the models 1 and 3 are greater than the critical energy in the model 2.

If we look at the soliton as a point particle, we can find the critical velocity by comparing the soliton energy when it

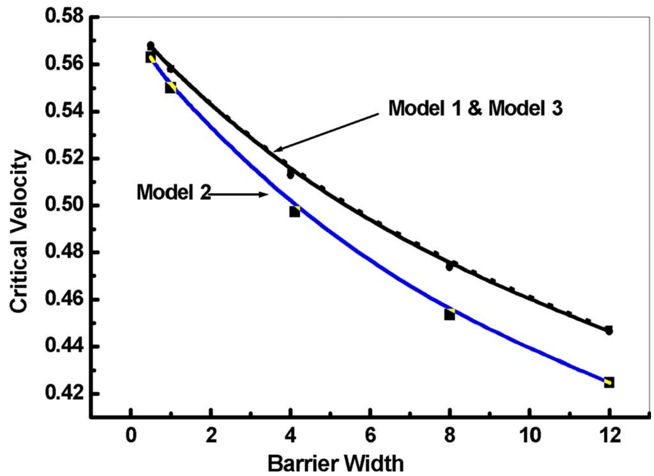

FIG. 3. (Color online) Critical velocity as a function of the barrier width. The potential is $v(x)=0.5e^{-4x^2}$.

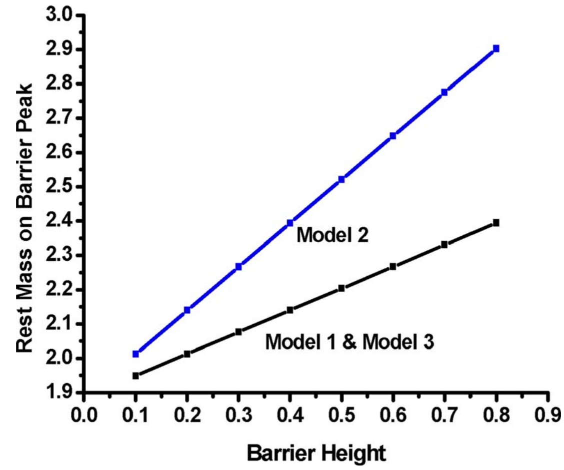

FIG. 4. (Color online) Rest mass of a static soliton on top of the barrier $v(x)=ae^{-4x^2}$ as a function of the height of the barrier

is located at the infinity and its energy at the top of the barrier. In this manner, we expect to find a greater critical velocity using the model 2 because the static energy (rest mass) on top of the barrier calculated with the model 2 is greater than the static energy (rest mass) in the other models. But the critical velocity in the model 2 is smaller than the critical velocity of the other models (see Figs. 2 and 3). It seems that it is in contradiction with the results of Fig. 1. Note that Fig. 1 presents the energy of a static soliton while the critical velocity is a dynamical parameter. An explanation in base of the effective mass has been presented in Ref. [4]. The reasoning can be completed if we inspect the problem by the collective coordinate approach which has been used for the sine-Gordon model previously [7]. The critical velocity is minimum required velocity for a soliton at the initial position of infinity in which the soliton is able to pass over the barrier after the interaction. The soliton energy in the position $X(t)=x_0-ut$ is $E[X(t)]=\int_{-\infty}^{+\infty}\mathcal{H}[x,X(t)]dx=\frac{M}{\sqrt{1-u^2}}$ where M is the soliton rest mass. The Hamiltonian density of each model is calculated by inserting the solution 4 in the Eqs. (9), (14), and (15) for the models 2, 3, and 1 respectively. Figure 4 presents the rest mass of the soliton as a function of the barrier height. This figure shows that the rest mass of the soliton in the model 2 is greater than the soliton rest mass in the models 1 and 3. Also models 1 and 3 predict almost the same rest mass for the soliton. It is clear that a soliton with a greater rest mass needs smaller velocity to reach the potential peak. This is true for small potentials as we can find in Fig. 2. But for greater potentials the differences between the models 1 and 3 become bigger. It is not surprising because the models are different. But it is possible to fit the results of the models with the definition of an effective potential [7].

Scattering of topological solitons on a potential well is more interesting. Unlike a classical point particle which always transmits through a potential well, a soliton may be trapped in a potential well with enough depth. We have found from Figs. 1 and 4 that a soliton has not a fixed mass during the interaction with a potential. So we cannot look at the soliton as a point particle in some cases. Several simulations have been done for soliton-well system using three models. Like the potential barrier, the general behavior of the system is almost the same for all three models. But there are some differences in details of the interactions. Figure 5 presents a comparison between the shapes of the potential well $v(x)=-0.5e^{-4x^2}$ as seen by the soliton in three models. Models 1 and 3 provide very similar potential but the shape of the potential in model 2 is different.

The differences in the effective potential for the three models cause some differences in the characteristics of the system. For example, the rest mass of the soliton is different when it is calculated using different models. Figure 6 demonstrates the rest mass of the soliton in three models as a function of the potential depth. Models 1 and 3 predict very similar values which are a little greater than the effective mass of the soliton in the model 2. Figures 4 and 6 depicting the rest mass for the cases of a barrier and a well respectively look related through a symmetry relation. The rest mass depends on the kinetic energy of a moving soliton, static energy of the potential $v(x)$ and also soliton static energy. The rest mass is calculated with integration of Hamiltonian density

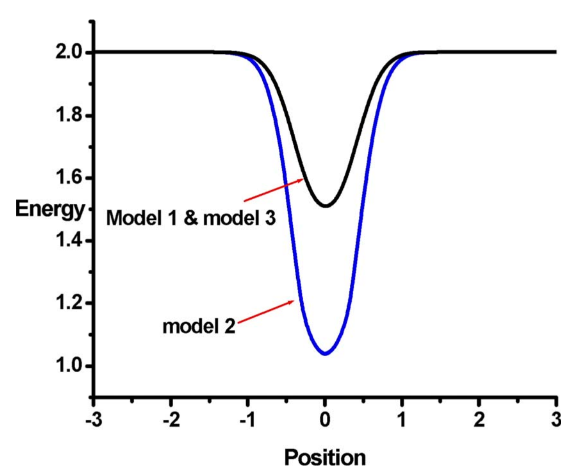

FIG. 5. (Color online) The shape of the potential well as seen by the soliton for the potential $v(x)=-0.5e^{-4x^2}$.

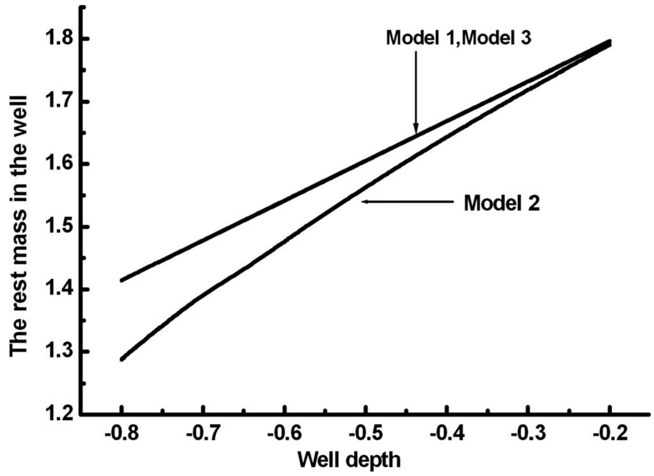

FIG. 6. The soliton rest mass in the potential well calculated with three models.

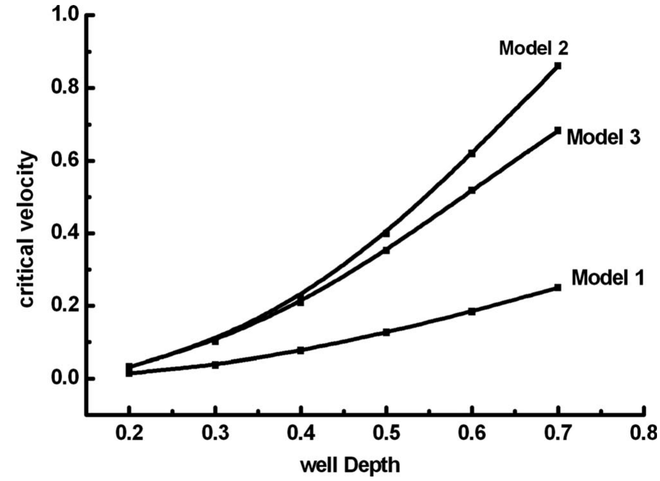

FIG. 7. velocity as a function of the depth of the potential well.

[Eq. (9) for model 2, Eq. (14) for model 3, and Eq. (15) for model 1] respect to the position "x." For a weak potential (small barrier or shallow well) the potential term is negligible. Therefore in the limit $v(x)=0$, Hamiltonian density in all three models become equal to each other. When the barrier height (in Fig. 4) or well depth (in Fig. 6) goes to zero, differences between the calculated rest mass in three models reduces. In the opposite way, the differences between calculated rest mass in three models increase when the strength of the potential increased.

The static part of Hamiltonian density in models 1 and 3 are completely equal. Also they are different from the static part of Hamiltonian density in model 2. Therefore the calculated rest mass using models 1 and 3 are very near to each other while they are different from the rest mass in the model 2. On the other hand, the Hamiltonian density in the model 2 gives bigger (smaller) energy density in comparison with the models 1 and 3 during the interaction with a potential barrier (well). This feature is reflected in Fig. 4 and Fig. 6 clearly.

As Fig. 5 shows, the effective potential in the models 1 and 3 are greater than the effective potential calculated with model 2. On the other hand, Fig. 6 indicates that the rest mass of the soliton in the model 2 is less than the rest mass in the other models. Therefore it is expected that the critical velocity of a soliton in the model 2 becomes greater than the critical velocity in the models 1 and 3. Figure 7 presents the critical velocity of a soliton in a potential well as a function of the potential depth. However the potential in the models 1 and 3 is almost the same, but the critical velocity in these models are not equal.

It is interesting to compare the soliton trajectory in three models. Figure 8 presents the trajectory of a soliton with an initial velocity $u=0.4$ during the interaction with potential $v(x)=-0.5e^{-4x^2}$. The final velocity of the soliton in the model 2 is smaller than the other two models. This means that the energy loss due to radiation in the model 2 is much greater than the models 1 and 3. Figure 8 also shows that the energy radiation in the model 3 is greater than the model 1. It is the main reason for the differences between the critical velocities in Fig. 7.

Critical velocity is a dynamical parameter. A soliton can pass over a potential (barrier or well) if it has enough kinetic energy in comparison with static energy. For a potential barrier situation, model 2 predicts bigger kinetic energy than the other two models. Therefore a soliton in model 2 needs smaller velocity to pass the barrier, as Fig. 2 shows. Differences in the kinetic energy of a soliton in the models 1 and 3 are very small and we cannot clearly compare the effects of parts of the energy in the soliton behavior. On the other hand in a potential well situation model 1 demonstrates bigger kinetic energy than the other models, thus a soliton in this model needs smaller velocity to pass the well as we can see in Fig. 7.

The most interesting behavior of a soliton during the scattering on a potential well is seen in some very narrow windows of initial velocities. At some velocities smaller than the u_c the soliton may reflect back or transmit over the potential while one would expect that the soliton should be trapped in the potential well. These narrow windows can be found by

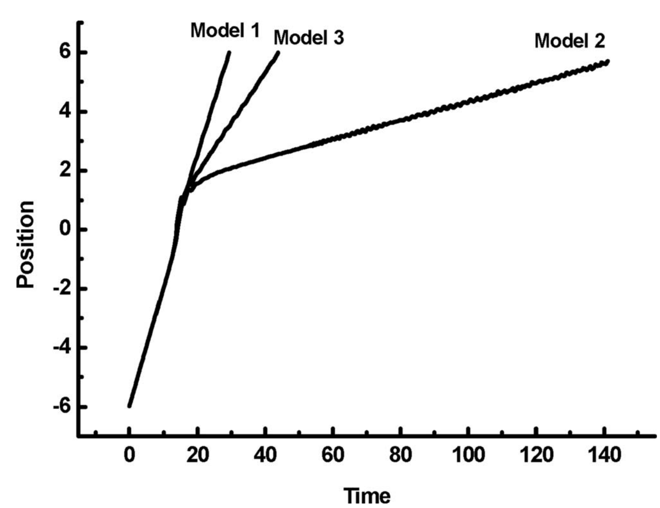

FIG. 8. Trajectory of a soliton with initial velocity $u=0.4$ during the interaction with the potential well $v(x)=-0.5e^{-4x^2}$ in three models.

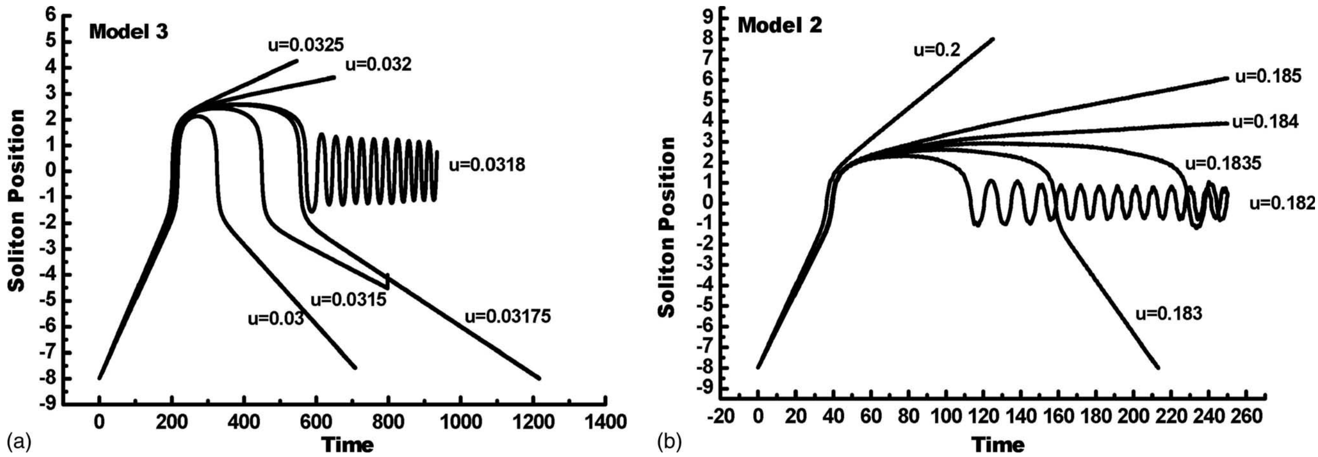

FIG. 9. (a) Soliton reflection from the potential well $v(x)=-0.2e^{-4x^2}$ in model 3. (b) Soliton reflection from the potential well $v(x)=-0.6e^{-4x^2}$ using the model 2.

scanning the soliton initial velocity with small steps. Figure 9 shows this phenomenon in the models 1 and 3. Figure 9(a) shows that a soliton with an initial velocity within the window $0.03 \leq u_i \leq 0.03175$ reflects back during the interaction with potential $v(x)=-0.2e^{-4x^2}$ simulated using the model 3. Figure 9(b) presents the same phenomenon as in model 2 for the potential $v(x)=-0.6e^{-4x^2}$. Same situations have not been reported in [6] for the model 1. We have setup some simulations for investigating this phenomenon in the model 1. Our simulations could not strongly confirm the existences of this situation in the model 1. Models 2 and 3 are built by varying the Lagrangian density with respect to both "field" and the "metric." Therefore energy exchange between the field and the potential in these models is possible. Soliton reflection in a potential well is a result of energy exchange between the soliton and the potential [1]. It is shown that the chaotic behavior of such solitary wave collisions depends on the transfer of energy to a second party (here the potential well) [9]. This behavior also has been observed for solitons of other models [3,10]. This phenomenon needs deeper investigations.

V. CONCLUSION AND REMARKS

The presented model is compared with other two models. The results of the interaction of a soliton with potentials using three models are almost in agreement with the other models. Model 1 adds the potential to the equation of motion by a different method from those that are used in models 2 and 3. Model 3 has been presented with adding an extra term in the metric of background space time. Added term in model 3 improves the approximation of potential effects on the behavior of soliton during the interaction with potential. On the other hand model 3 predicts the characteristics of the system very near to predictions of the model 1. Therefore it can be concluded that the results of the models are valid. The model 1 is suitable for using in numerical simulations while model 3 is more analytic. There are several features which are similar in three models. Therefore they are model independent. All three models agree that the interaction of a soliton with a potential barrier is nearly elastic. At low velocities it reflects back but with a high velocity climbs the barrier and transmits over the potential. There exists a critical velocity which separates these two kinds of trajectories. It is possible to equalize the soliton trajectory and critical velocity in three models with defining effective parameters for the models. This procedure has been done for models 1 and 2 on sine-Gordon solitons in [7] successfully. Soliton radiates some amounts of energy during the interaction with the potential. The amount of radiated energy is not the same in three models and it depends on the selected model. Interaction of a soliton with potential well is more inelastic. It is possible that a soliton scatters on a potential well and reflects back from the potential. This phenomenon is strongly model dependent. Solitons in the models 2 and 3 are able to exchange energy with potential well. Therefore they may find enough energy from the potential and escape from the well. In conclusion we suggest model 3 for studying the soliton-potential systems. This model contains almost all the common features of the models 1 and 2.

It is interesting to study the soliton-potential system using collective coordinate method with model 3. The constructed collecting coordinates for such system gives us better information about the soliton dynamics.

Scattering of the solitons of the other models on defects using model 3 can be investigated too. These studies help us to improve our knowledge about the general behavior of solitons.

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